11th International Conference on New Ideas in
MANAGEMENT, ECONOMICS & ACCOUNTING
17-19 February 2023 Barcelona, Spain

The Dynamic Portfolio - The Minimum Risk and The Maximum Growth

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Abstract

This paper deals with portfolio selection, especially from the viewpoint of dynamics. Markets composed of many agents, such as the security market and the currency market, feature price fluctuations over time, which are often caused by changes in anticipations about the future. Investors have considered this as unfavorable, since it entails risk. To deal with this, Markowitz contrived a static portfolio selection theory, and that theory has now been developed with a dynamic principle. We also follow this trend: Based on the Wiener process, we establish a dynamic portfolio and find two important issues that involve two crucial phenomena which coexist on an efficient frontier: one is a situation with minimum risk; the other is one with maximum growth rate of the portfolio. In the latter case, the price fluctuations are not unfavorable but desirable. The two can coexist, because they are on the same efficient frontier curve. Furthermore, the equal weight of assets included in the portfolio entails a situation with minimum risk. This idea has been adopted in mutual funds, but we demonstrate preconditions for this idea. Moreover, the portfolio with the maximum growth rate is determined only by risk assets. Our discussions are made based on rigorous mathematical calculations.

Keywords: efficient frontier; maximum return point; minimum risk point; portfolio selection; Wiener process

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1. Introduction

Portfolio selection is commonplace but exciting. The excitement goes back to Markowitz (1952) and its innovative ideas, one of which is that initially, investors need to choose promising assets by some standards, before making portfolio selections; After that, idealistically, the temporal series of returns from an asset should be considered. And probability distributions of returns are a function with an argument regarding time; these ideas imply that Markowitz recognized the importance of the fluctuations of asset values over time. Markowitz also took up the issue of doubt concerning whether the analyses based on mean-variance can be unconditionally compatible with the utility maximization principle of investors or not.

Later, many researchers such as Samuelson (1969) and Luenberger (1993) discussed with the latter issue. And as for the former issue, lots of researchers, such as Merton (1973) and Duffie (2001), examine how asset prices randomly move and how optimal portfolios are determined. In this context, a key technique is stochastic analysis involving Brownian motion and the Wiener process. Duffie (2001) offers comprehensive surveys, in which the Wiener process is argued as Brownian motions, and Ito process is discussed as a developed version from the Wiener process. And a key concept is dynamic portfolio building, unlike the static approach found in the Markowitz, though that continues to be a main pillar in the field.

Motivated thus, we focus on dynamic asset pricing in terms of a portfolio. Specifically, from the viewpoint of the maximization of the expected growth rate of a portfolio, we establish an optimal dynamic portfolio, in which we posit two findings: first, that there is a maximum point on an efficient frontier; second, that giving equal weight to the assets comprising a portfolio entails minimum risk. We rigorously analyze our arguments. Additionally, we focus our discussions on a portfolio with 3 risk assets, for simplicity.

In the next section, we offer our stochastic model regarding individual price movement of risk assets, which is a baseline for subsequent discussions. Subsequently, section 3 demonstrates necessary conditions for the maximization, and in section 4 we derive an efficient frontier, then prove the existence of the maximum and minimum points on the frontier curve. In what follows, section 5 is devoted to detailed arguments about equal weights and the minimum variance. Finally, section 6 is devoted to concluding remarks.

2. Dynamic Portfolio

We assume that there are n risky assets, whose prices follow normal geometric Brownian motions. The price of the asset i $(i = 1,2,\ldots n)$ can be described as

$$\frac{\mathrm{dp}_i(t)}{p_i(t)} = \mu_i(t)dt + \mathrm{dz}_i.$$

Here, z_i shows the Wiener process with a variation σ_i^2. We describe price changes in terms of continuous time. So, the Wiener process is shown in the above. Generally, the process describes random movements between two different time. Then, it follows a normal distribution with a mean and variation. In line with this, the above expression is deduced. In general, all assets have correlations, which are described as $\mathrm{cov}(\mathrm{dz}_i, \mathrm{dz}_j) = E(\mathrm{dz}_i, \mathrm{dz}_j) = \sigma_{\mathrm{ij}}dt$. Additionally, we define a covariance matrix supposed to be regular, as $S = [\sigma_{\mathrm{ij}}]$, (j = 1,2,...,n).

When we consider a portfolio composed of n risky assets, which is valued by weights, wi. The weights have a sum is 1. Then, we have Theorem 1, as follows:

Theorem 1 *The instant return rate of a portfolio equals the weighted sum of its individual assets.*

Proof 1 *The value of a portfolio V is defined as a product of individual assets prices, such as* $V = p_1p_2 \ldots p_n$, *which can be transformed as* $\frac{\mathrm{dV}}{V} = \frac{\mathrm{dp}_1}{p_1} + \frac{\mathrm{dp}_2}{p_2} + \cdots + \frac{\mathrm{dp}_n}{p_n}$. *The above relationship implies that the weighted coefficients,* w_i, *are defined in relation to the instant change rate in the prices of the individual assets.* Thus, we establish

$$\frac{\mathrm{dV}}{V} = \sum_{i=1}^{n} \frac{\mathrm{dp}_i w_i}{p_i} = \sum_{i=1}^{n} (\mathrm{dt}\mu_i w_i + \mathrm{dz}_i w_i).$$

Based on the above analyses, the value V (t) obviously follows a normal distribution. So, logV (t) follows a log normal distribution, which has properties such as

$E\left[Log\left(\frac{\mathrm{v}(t)}{v(0)}\right)\right] = \mathrm{vt} = \sum_{i=1}^{n} t\, \mu_i w_i - \frac{1}{2}\sum_{i,j=1}^{n} t\, w_i \sigma_{\mathrm{ij}} w_j$ and $\mathrm{Var}\left[Log\left(\frac{\mathrm{v}(t)}{v(0)}\right)\right] = \sum_{i,j=1}^{n} t\, w_i \sigma_{\mathrm{ij}} w_j$.

3. The Optimal Portfolio Structure

We explore the weights w_i to maximize the expectations, keeping the variance of the portfolio constant. In line with this, we establish the formulas such as

$$\max\left(\sum_{i=1}^{n} \mu_i w_i - \frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j\right), \qquad (1)$$

$$\sum_{i=1}^{n} w_i = 1, \qquad (2)$$

and

$$W^T SW = \sigma^2, \qquad (3)$$

where σ^2 shows the variance of the portfolio, and W^T means the transposed vector of W. Below, we define n = 3, and denote symbols as $W = (w_1, w_2, w_3)',\ U = (\mu_1, \mu_2, \mu_3)'\ and\ I = (1,1,1)'$. In conditions of (2) and (3), we seek to execute (1). In these processes, we establish a Lagrangian function as

$$L = W^T U - \frac{1}{2}W^T SW - \lambda(W^T I - 1) - \frac{1}{2}\xi(W^T SW - \sigma^2).$$

Here, λ and ξ are Lagrangian multipliers. Then, we obtain the necessary conditions for maximizing L as follows: First, partially differentiating L with w_1, w_2, and w_3, the next relationship prevails.

$$\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} + \xi\begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} + \begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix}$$

Second, based on (2) and (3), the system is composed of 5 variables, such as w_i, λ , ξ , and 5 equations. For simplicity, we assume as follows: The co-variances among the assets are null, and the variances of the 3 assets are common, $\beta > 0$. Noticing $\sigma_{ij} = \sigma_{ji}$, we can obtain a simpler relationship, in which λ and ξ are eliminated, such as

$$(\mu_2 - \mu_1)(w_1 - w_3) + (\mu_1 - \mu_3)(w_1 - w_2) = 0, \qquad (4)$$

which can be transformed as

$$\frac{w_1 - w_2}{w_1 - w_3} = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3}. \qquad (5)$$

Next, let us assume that w_1 is constant. Then, we can conjecture (5) such that the ratio of $w_1 - w_2$ to $w_1 - w_3$ is equal to that of $\mu_1 - \mu_2$ to $\mu_1 - \mu_3$. That is, the weight of asset 1 is determined as a center, and the weights of assets 2 and assets 3 are respectively determined above and below w_1 at an equal distance. This structure applies to the 3 asset expectations, specifically, the relationship between them. Later, we will explain this conjecture, in section 4 in Figure 2. Expectation is a key factor in the determination of the weights. Thus, our system comprises 3 simultaneous equations as seen in (4), and 2 constraints.

4. Complete Diversification, Equal Weights, and Minimum Dispersion

Based on the discussion so far, our system is summarized as

$$w_1 + w_2 + w_3 = 1,$$
$$\sigma^2 - \beta(w_1^2 + w_2^2 + w_3^2) = 0,$$

and (4). Solving the system in relation to w_i, we obtain the following solution:

$$w_1 = \frac{1}{3}, \qquad (6)$$

$$w_2 = \frac{1}{3} - \frac{1}{\sqrt{6}}\sqrt{\left(\frac{3}{\beta}\right)\sigma^2 - 1}, \qquad (7)$$

$$w_3 = \frac{1}{3} + \frac{1}{\sqrt{6}}\sqrt{\left(\frac{3}{\beta}\right)\sigma^2 - 1}, \qquad (8)$$

where (7) and (8) are alternative relationships.

At this stage, we need to elucidate why (6) appears as a solution. We postulate that (5) equals -1, which refers to $\mu_1 = \frac{\mu_2 + \mu_3}{2}$. The expectation of asset 1 equals the weighted average of the expectations of asset 2 and asset 3. This relationship leads to $w_1 = \frac{w_2 + w_3}{2}$. Then, (6) prevails. Therefore, we can obtain (7) and (8) as the solution of the system, in which $w_1 = 1/3$.

Initially let us analyze (7), which obviously implies $\sigma^2 \geq \left(\frac{\beta}{3}\right)$.

This shows that the minimum variation of the portfolio equals to the value of individual asset variation divided by the number of the assets. This property shows that the risks of the portfolio can be greatly reduced by increasing the number of assets included in it. In brief, w_2 and w_3 are determined by a choice regarding σ by investors. But we can say that if investors choose the minimum risk, that is, the minimum variance, the two are determined 1/3. Thus, we obtain Theorem 2:

Theorem 2 *If investors choose equally weighted on the assets, the portfolio attains the minimum risk, and verse versa.*

Proof 2 *Substitute* $\sigma^2 = \beta/3$ *into (7) and (8), and* $w_2 = w_3 = {}^1/_3$ *prevails On the other hand, if investors choose 1/3 in (7) and (8),* $\sigma^2 = \beta/3$ *holds.*

Next, we proceed to explore an efficient frontier. Here, we can define the expected maximum growth rate function as

$$\Lambda = -\frac{\beta}{2}(w_1^2 + w_2^2 + w_3^2) + \mu_1 w_1 + \mu_2 w_2 + \mu_3 w_3. \tag{9}$$

Then, substituting (6), (7) and (8) into (9), we derive a function with an argument σ as

$$V_1(\sigma) = \frac{(\mu_3 - \mu_2)\sqrt{\left(\frac{3}{\beta}\right)\sigma^2 - 1}}{\sqrt{6}} + \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) - \frac{\sigma^2}{2}. \tag{10}$$

When $\mu_3 - \mu_2 > 0,$ the above formula shows the efficient frontier curve that we seek. The curve offers two important types of information from the viewpoint of the optimal choices of investors: one is a determination of the highest growth rates; the other the minimum variation. In this situation, we elucidate (10) more intensively.

Differentiating and analyzing (10), we find the standard variation, σ_m, which refers to the maximum growth rate of the portfolio. σ_m is described as

$$\sigma_m = \sqrt{\frac{2\beta^2 + 3(\mu_3 - \mu_2)^2}{6\beta}},$$

and the expected maximum growth rate of the portfolio is revealed. In short, when $\sqrt{\frac{\beta}{3}} < \sigma \leq \sigma_m$, $\frac{dV_1}{d\sigma} \geq 0$, while $\frac{dV_1}{d\sigma} < 0$, when $\sigma > \sigma_m$. Therefore, the efficient frontier curve is $V_1(\sigma)$, corresponding to a range of $\sqrt{\frac{\beta}{3}} < \sigma \leq \sigma_m$.

This finding of $\sigma_m$without no-risk assets is important. In our context, the combinations of risky assets are determined, while in Markovich (1952) the determination is of non-risk assets.

The second finding is clear. At the minimum variation, the 3 weights are obviously equal to 1/3, as mentioned in Theorem 2.

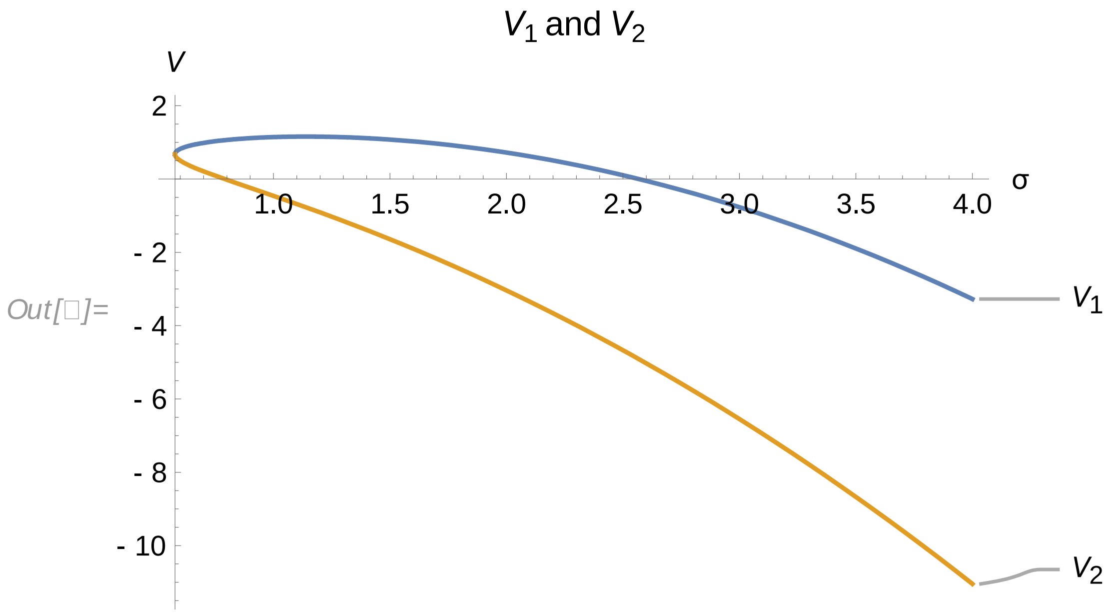
Furthermore, let us analyze the differences between (7) and (8). Substituting w_3 and w_2 respectively instead of w_2 and w_3 into (9), we derive lower frontier V_2 as

$$V_2(\sigma) = \frac{(\mu_2 - \mu_3)\sqrt{\left(\frac{3}{\beta}\right)\sigma^2 - 1}}{\sqrt{6}} + \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) - \frac{\sigma^2}{2}. \tag{11}$$

Notice that as an investment possibility set, V_1 shows an upper boundary, in which the efficient frontier is included, while V_2 shows a lower boundary. Since we have discussed the properties concerning the upper boundary, we need to examine the lower one. But here we omit it in the interest of space. See Figure 1.

The efficient frontier has two important properties: one is the minimum risk point; the other the highest expected price growth rate. Let us depict and verify this logic. We assume: μ_1 = 0.842; μ_2= 0.148; μ_3 = 1.536; β = 1. Then, we can describe a possible investment set in Figure 1.

Figure 1: The possible investment set: on this curve, the minimum dispersion point and the maximum growth rate point coexist.



Source: my own research

The possible investment set is expressed in a gray area, and the partial upper frontier of the area is an efficient frontier. Specifically, only the part of an upper border in a range corresponding to $\sqrt{\frac{1}{3}} \leq \sigma \leq \sigma_m$ is an efficient frontier, because in these regions, if investors prefer higher growth rates for the portfolio, the corresponding larger risks are reasonable at cost. Briefly, in these ranges, the choice of higher growth rates and higher risks is rewarded. This is why we focus on stochastic asset markets, in which the growth rates of the returns are depressed, which is shown in the third term on the right side in (10), i.e. $-\sigma^2/2$. This term does not reflect the risk aversions of investors but preferable fluctuations instead. However, in $\sigma > \sigma_m$, investors face a choice with small growth rates and larger risks, which is contradictory for reasonable investors.

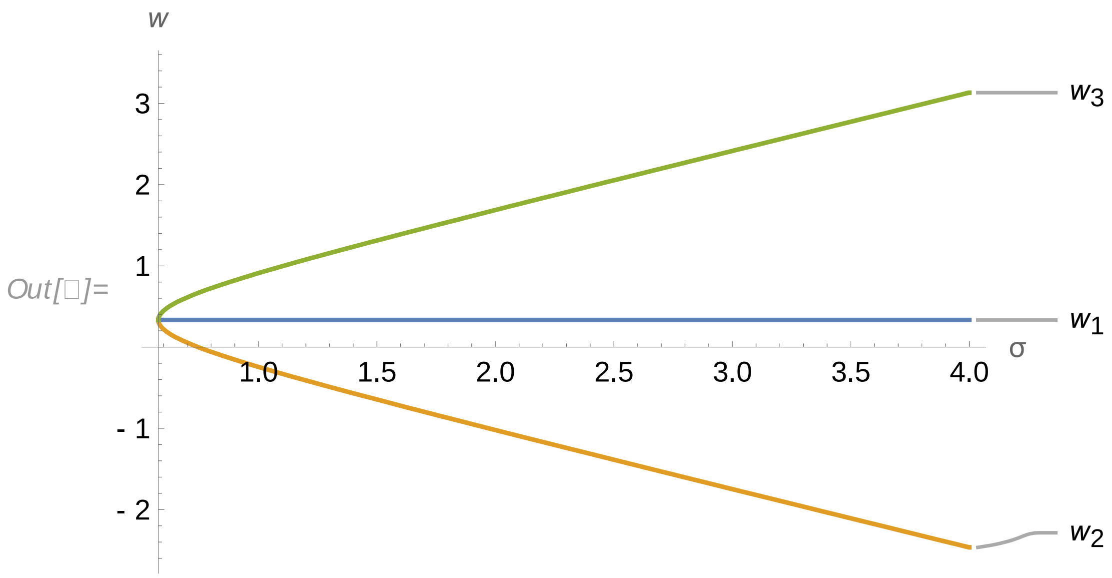
Moreover, this phenomenon does not appear in Markowitz (1952), which deals with risk aversion in one period. The difference between the two is the time period. In a one- period model, factors like σ^2, which can positively affect expected returns, do not appear. In this sense, this point is crucial, since our model shows the maximum expected growth rate.

At this stage, we should intensively investigate the weights, which have been established in (6), (7) and (8).

One weight is 1/3 regardless of σ, and the others are located on the upper and lower sides of 1/3 at equal distance, and the two distances from 1/3 keep constant as β moves. Then, these relative locations are indifferent in relation to β. This mechanism is a key to balance the weights totally. We can show the above logics in Figure 2.

Concerning the second asset, we should be careful: In some cases, it can be shorted or can not be included in the portfolio. The former holds in the case of $w_2 < 0$, which results in $\sigma \geq \frac{\sqrt{5\,\beta}}{3}$, while the latter holds in the case of $w_2 = 0$, which results in $\sigma = \frac{\sqrt{5\,\beta}}{3}$. Thus, this portfolio can include short sales and be considered as having only two assets.

Figure 2: The three weights: one is 1/3; the others are located at a similar distance from 1/3.



Source: my own research

5. Equal weights and a minimum dispersion

Judging from Figure 2, when the 3 weights are equal, 1/3, minimum variance seems to prevail. This is not necessarily obvious. In our context, in section 4, Proof 2 does not logically argue $\sigma^2 = \frac{\beta}{3}$ as a minimum variance. So, we need to discuss this issue.

Obviously, if $\sigma = \sqrt{\frac{\beta}{3}}$, $w_i = \frac{1}{3}$ prevails, and the corresponding minimum growth rate is

$$V_1\left(\sqrt{\frac{\beta}{3}}\right) = V_2\left(\sqrt{\frac{\beta}{3}}\right) = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3) - \frac{\beta}{6}.$$

In general, (7) and (8) are alternatives, but at $\sigma = \sqrt{\frac{\beta}{3}},$ the two have a common point. In this situation, $\sigma^2 = \frac{\beta}{3}$ seems to be the minimum variance. Here, we prove it. Thus, our system can be expressed by the following 3 simultaneous equation system:

$$w_1^2 + w_2^2 + w_3^2 = \frac{\sigma^2}{\beta} \qquad (12)$$

$$w_1 + w_2 + w_3 = 1,$$

$$w_1 = \frac{w_2 + w_3}{2}.$$

Based on the last 2 equations, $w_1 = 1/3$ is derived. Then, our system is reduced to (12) and $w_2 + w_3 = 2/3$. Solving this system concerning w_2 in order to minimize $(\frac{\sigma^2}{\beta})$, we can find the minimum $(\frac{\sigma^2}{\beta})$ to be 1/3, which implies that the minimum variance of the portfolio is $(\frac{\beta}{3})$. In brief, if investors choose the equal weights 1/3, they opt for minimum diversion. On the other hand, if they choose a minimum dispersion on the efficient frontier, they evenly allot their weights of the assets.

6. Concluding Remarks

We research how to construct a suitable portfolio from a dynamic viewpoint, in which the values are randomly fluctuating over time. Using dynamic portfolio selection theories developed after the advent of the pioneering statistical one in Markovich (1952), we focus on the growth rate of a portfolio and naturally establish a model concerning how to discover suitable weights imposed on individual assets. In this context, we basically utilize the Wieners processes, which follows a normal distribution and has properties of a mean and variations. As a result, the expected growth rate of returns is affected by fluctuations. These fluctuations curb the expected growth of the value of the portfolio and eventually reverse a tendency from high risk with high return to high risk with low return. This implies the existence of the maximum expected growth rate and the variation of the portfolio corresponding to the maximum. Moreover, we prove that when investors acquire assets with equal weights, the resulting portfolio is at the minimum risk.

Our conclusion that equal weights necessarily entail the minimum risk has verifiable, because many mutual funds really adopt this strategy. However, they have to recognize the pre-condition that the assets comprising of a portfolio should have no correlations.

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