Binary Sequences with Large SRS Contrast Ratios

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Abstract

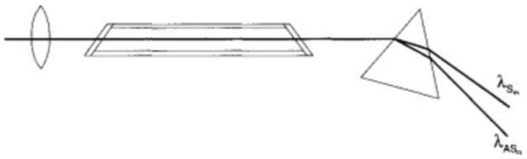
Stimulated Raman Scattering (SRS) is a very useful tool to extend the frequency range of fixed-frequency and tunable lasers. In particular, In the use of laser pulses to control chemical reactions, a successful management technique called pseudorandom binary phase shaping (BPS) have been used to achieve much cleaner coherent excitation of a Raman mode at certain frequency. A core technical problem in BPS is to find pseudorandom binary sequences with large SRS contrast ratios. In the past, with aid of more and more powerful computers, the main search methods for such 'ideal' sequences were conducting exhaustive search within a certain scope narrowed by experiences and some mathematical knowledge such as number theory. In this paper, in the light from the known results in Merit Factor Problem, we have given several families of various lengths that have large asymptotic SRS contrast ratios. Our results have shown that Jacobi family Sequences, are still the best candidates to obtain binary sequences with large SRS contrast ratios. Our numerical analysis has shown that the SRS contrast ratios of the sequences investigated in this paper, are significantly larger than the results in existing documents.

Keywords: SRS, Pseudorandom sequences, Contrast Ratio

1. Introduction

Stimulated Raman Scattering (SRS) is a very useful tool to extend the frequency range of fixed-frequency and tunable lasers. Figure 1 is a concrete example of accomplishing SRS by focusin a laser beam onto a nonlinear medium, to generate emission at a series of wavelengths

Figure 1: Optical configuration for H_2 Raman shifter ([1], Figure 8.9).



Given a laser pulse, described by having intensity $E(\omega)$ at frequency ω, we are interested in the intensities that result from two types of nonlinear, second order interference. In the case of impulsive SRS, the interference is negative in that the second order intensity at frequency ω is given by $E^{SRS}(\omega) = \int_{\alpha-\beta=\omega} E(\alpha)E^{*}(\beta)\, d\alpha$.

In binary phase shaping the situation is simplified by considering a pulse that only contains a finite number of equally spaced frequencies (indexed by $0 \leq k \leq n-1$) with each frequency admitting only a limited set of intensities. Specifically, a given frequency can be masked to 0 but otherwise it is at a fixed level, the only choice allowed being whether the phase is left unchanged or is switched by π. The shape of a given laser pulse can thus be encoded by the shaping sequence $\boldsymbol{e} = (e_i | 0 \leq i \leq n-1)$ where $e_i = 0$ if frequency *i* has been masked out, and otherwise e_i is +1 or -1 depending on whether the phase change is 0 or π. In this case the above intensity measures at the *k*'th frequency become $e^{SRS}(k) = \sum_{a-b=k} e_a e_b$

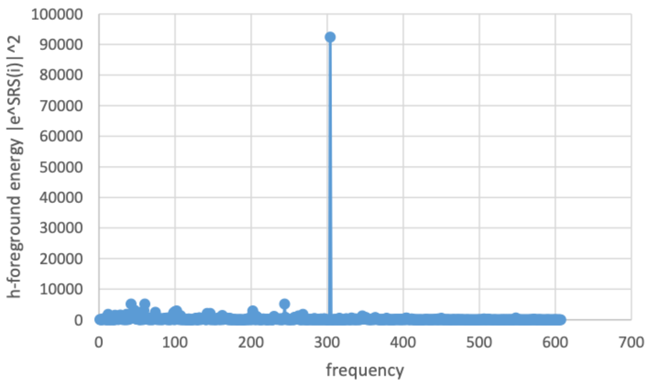
Our present goal is to find shaping sequences $\boldsymbol{e}$ that allow focus at a chosen frequency. That is, for SRS we seek sequences for which the energy $|e^{SRS}(h)|^2$ is large at the desired focus frequency h while the remaining energies $|e^{SRS}(k)|^2$ are small for all k≠h. Then for a chosen focus frequency h we wish to choose a shaping sequence $\boldsymbol{e}$ to accomplish various things:

1) maximize the h-foreground energy $|e^{SRS}(h)|^2$;
2) minimize the h-background energy $b^{SRS}(h)$=$t^{SRS} - |e^{SRS}(h)|^2$, where $t^{SRS} = \sum_{k=1}^{n-1} |e^{SRS}(k)|^2$
3)maximize the h -contrast ratio $CR^{SRS}(h)$=$\frac{|e^{SRS}(h)|^2}{b^{SRS}(h)}$,

or, equivalently, the h-signal-to-noise ratio $\frac{S}{N_R(h)} = \frac{(n-2)\left|e^{SRS}(h)\right|^2}{b^{SRS}(h)}$

Although theoretically, the focus energy h can take any value in [1,n-1], scientists have commonly chosen $h = \left\lfloor \frac{n}{2} \right\rfloor$ ([2],[3]). In this case, it is natural to construct the sequence $\boldsymbol{e}$ in a ``symmetric" form: $\boldsymbol{e} = (c \mid \pm c)$. For instance, in [2], Lozovoy, Xu, Shane and Dantus constructed a sequence c of length 304. so $\boldsymbol{e}$ has length *n=608*. Then the focus $\left\lfloor\frac{608}{2}\right\rfloor$ =304 foreground energy $|e^{SRS}(304)|^2$ takes the maximum value 304^2=92416, the contrast ratio $CR^{SRS}(304)$=0.6557, and $\frac{S}{N_R(304)}$=397.37, as shown in Figure 2. Dantus has informed us that the S/N is the most commonly used definition for signal to background ratio.

Figure 2: Lozovoy's team has found a binary symmetric sequence with length 608 with the maximum $|e^{SRS}(304)|^2$ *=92416, the contrast ratio* $CR^{SRS}(304)$*=0.6557 and* $\frac{S}{N_R(304)}$*=397.37*



2. $\boldsymbol{CR^{SRS}}$ contrast ratio for sequences $\boldsymbol{e} = (\boldsymbol{c} \mid \boldsymbol{c})$.

The following notations will be heavily used in this paper.

Let $\boldsymbol{x} = (x_0, x_1, \cdots, x_{N-1})$ be a binary sequences of length *N*, where $x_i = \pm 1$. The aperiodic autocorrelation function of $\boldsymbol{x}$ at shift *i* is defined to be

$$A_{\boldsymbol{x}}(i) = \sum_{j=0}^{N-i-1} x_j x_{j+i}, \ \ i = 1, \ldots, N-1 \qquad (1)$$

And we also need the ``alternate" aperiodic autocorrelation function the same sequence of $\boldsymbol{x}$ at shift *i* is defined to be

$$\widetilde{A_{\boldsymbol{x}}(i)} = \sum_{j=0}^{N-i-1} (-1)^j x_j x_{j+i}, \ \ i = 1, \ldots, N-1 \qquad (2)$$

The periodic autocorrelation function (PACF) of $\boldsymbol{x}$ at shift *i* is defined to be

$$P_{\boldsymbol{x}}(i) = \sum_{j=0}^{N-1} x_j x_{j+i}, \ \ i = 0, \ldots, N-1 \qquad (3)$$

In [4], Parker has defined the Negaaperiodic autocorrelation function (NACF) of sequence $\boldsymbol{x}$ at shift *i* as

$$Q_{\boldsymbol{x}}(i) = \sum_{j=0}^{N-1-i} x_j x_{j+i} - \sum_{j=N-i}^{N-1} x_j x_{j+i}, \ i = 0, \ldots, N-1 \qquad (4)$$

where all the subscripts in equations above are taken modulo *N*.

If the sequence $\boldsymbol{x}$ is binary, which means that all the x_j's are +1 or -1, the merit factor of the sequence $\boldsymbol{x}$, introduced by Golay [5], is defined as

$$F_{\boldsymbol{x}} = \frac{N^2}{\sum_i^{N-1} A_X^2(i)} \tag{5}$$

The following property describes a connection between the autocorrelations of sequences $\boldsymbol{b}$ and $\boldsymbol{c}$ which provides a foundation for further discussions. We omit the proofs to all the Theorems and properties provided in this paper. The proofs are available upon request.

Property 2.1 For a positive integer *m*, let $\boldsymbol{c} = (c_0, c_1, \cdots, c_{m-1})$ be a binary sequence of length *m*. If a sequence $\boldsymbol{e} = (c \mid \pm c)$. Then

$$e^{SRS}(k) = \begin{cases} M_c(k) + A_c(k), & if\ 1 \le k < m; \\ (-1)^\delta \times m, & if\ k = m; \\ (-1)^\delta \times A_c(k-m), & if\ m < k < 2m; \end{cases}$$

Where $\boldsymbol{M_c(k) = P_c(k)}$ and variable $\boldsymbol{\delta = 0}$ if $\boldsymbol{e = (c \mid\ c)}$; $\boldsymbol{M_c(k) = Q_c(k)}$ and variable $\boldsymbol{\delta = 1}$ if $\boldsymbol{e = (c \mid -c)}$;

The following property provides a foundation for the important results which will be introduced later.

Property 2.2 For a positive integer *m*, let $\boldsymbol{c} = (c_0, c_1, \cdots, c_{m-1})$ be a binary sequence of length *m*. If a sequence $\boldsymbol{e} = (c \mid \pm c)$. Then

$$\frac{1}{CR^{SRS}(m)} = \frac{1}{F_c} + \frac{\sum_{k=1}^{m-1} M_c^2(k)}{m^2} + 2 \times \frac{\sum_{k=1}^{m-1} M_c(k) A_c(k)}{m^2}$$

where F_c is the merit factor of sequence $\boldsymbol{c}$ as defined in expression (5); $M_c(k) = P_c(k)$ if $\boldsymbol{e} = (c \mid c)$, where $P_c(k)$ is the PACF of sequence $\boldsymbol{c}$ at shift *k* as defined in (3) or $M_c(k) = Q_c(k)$ if $\boldsymbol{e} = (c \mid -c)$, where $Q_c(k)$ is the NACF of sequence $\boldsymbol{c}$ at shift *k* as defined in (4).

The two Properties imply that for a fixed $\boldsymbol{m}$ value, in order to obtain a sequence with a large CR^{SRS} value, we need to look for binary sequences $\boldsymbol{c}$ of length $\boldsymbol{m}$ with high merit factor value F_c, and low PACF $P_c(k)$s, or low NACF $Q_c(k)$ s at all shifts *k*. Then we can construct sequence $\boldsymbol{e} = (c \mid \pm c)$, based on $\boldsymbol{c}$.

The first candidate for base sequence $\boldsymbol{c}$ is called Legendre sequences and more generally, Jacobi sequences. For readers convenience, we give a brief introduction here:

For *p* an odd prime, a Legendre sequence of length *p* is defined by the Legendre symbols

$\alpha_j = \left(\frac{j}{p}\right), j = 0, \cdots, p-1$, where $\left(\frac{j}{p}\right) = \begin{cases} 1, & if\ j\ is\ 0\ or\ a\ square\ modulo\ p; \\ -1, & otherwise \end{cases}$

Furthermore, for an odd number N, where $N = pq$ with $p < q$ distinct odd primes, a modified Jacobi sequence could be defined as in [6] and [7]. $\boldsymbol{\beta}$ is a binary sequence of length N is defined as

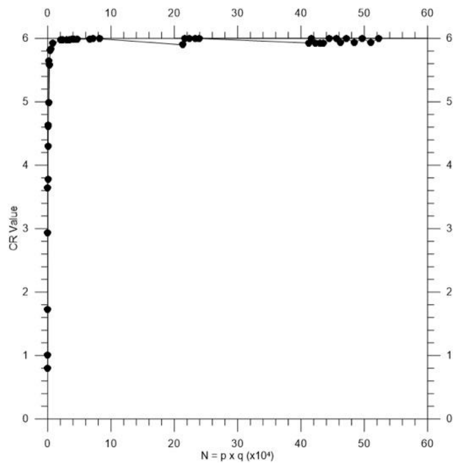
$$\beta_j = \left(\frac{j}{p}\right) \times \left(\frac{j}{q}\right), \;\; j = 0, \cdots, N-1$$

Note that if we could allow either p or q to be 1, then Legendre sequence is a special case of Jacobi sequence.

Now we are ready to state our first important result of this paper.

Theorem 2.3 For two different odd primes $p < q$, where p could be 1 or odd prime. Let $\boldsymbol{\alpha}$ be a Legendre sequence of length q, or a Jacobi sequence of length pq as defined before. Let $\boldsymbol{\alpha}^{\frac{1}{4}}$ be the sequence obtained by cyclically shifting $\boldsymbol{\alpha}$ by a ratio of $\frac{1}{4}$. Define sequence $\boldsymbol{e}$ of length *2p* or *2pq* as $\boldsymbol{e} = (\alpha^{\frac{1}{4}} \mid \alpha^{\frac{1}{4}})$. Then the asymptotic contrast ratio $CR^{SRS}(q)$ for Legendre or $CR^{SRS}(pq)$ for Jacobi is 6.0, when $p = 1$ or $p < q$ are distinct odd primes with p and q close enough.

Figure 3: The asymptotic central CR^{SRS} ratio of Jacobi sequences is 6.0, where p and q are adjacent odd primes



3. $\boldsymbol{CR^{SRS}}$ contrast ratio for sequences $\boldsymbol{e} = (\boldsymbol{c} \mid -\boldsymbol{c})$.

The following definition comes from [8].

Definition 3.1 Let $\boldsymbol{\epsilon^0} = (++--++--++--,\cdots)$, $\boldsymbol{\epsilon^1} = (+--++--++--+,\cdots)$, and for any binary sequence $\boldsymbol{c} = (\boldsymbol{c_0}, \boldsymbol{c_1}, \cdots, \boldsymbol{c_{N-1}})$ be a binary sequence of length N. define $\boldsymbol{c} * \boldsymbol{\epsilon^0} = (\boldsymbol{c_0}, \boldsymbol{c_1}, -\boldsymbol{c_2}, -\boldsymbol{c_3}, \boldsymbol{c_4}, \boldsymbol{c_5}, -\boldsymbol{c_6}, -\boldsymbol{c_7} \cdots,)$, and

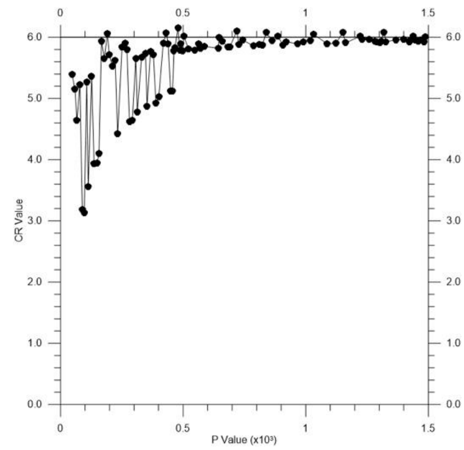
$\boldsymbol{c} * \boldsymbol{\epsilon^1} = (\boldsymbol{c_0}, -\boldsymbol{c_1}, -\boldsymbol{c_2}, \boldsymbol{c_3}, \boldsymbol{c_4}, -\boldsymbol{c_5}, -\boldsymbol{c_6}, \boldsymbol{c_7} \cdots,)$.

In particular, if $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{p-1})$ be a Legendre sequence of odd prime length p, let $\boldsymbol{\beta} = (\alpha \mid \alpha) * \epsilon^{\delta}$, where $\delta = 0, 1$. This type of sequence was first defined in [8] so we will call it TH sequence in the future.

Now we are ready to mention the second important result of this paper.

Theorem 3.2 For an odd prime number *p*, let $\boldsymbol{\alpha}$ be a Legendre sequence of length *p*. Let $\boldsymbol{\beta} = (\alpha \mid \alpha) * \epsilon^{\delta}$ be as defined as in Definition 3.1. Define a sequence $\boldsymbol{e} = (\boldsymbol{\beta} \mid -\boldsymbol{\beta})$. of length 4p. Then the asymptotic contrast ratio $CR^{SRS}(2p)$ is 6.0.

Figure 4: The asymptotic $CR^{SRS}(2p)$ ratio of TH Sequences of length 4p is 6.0



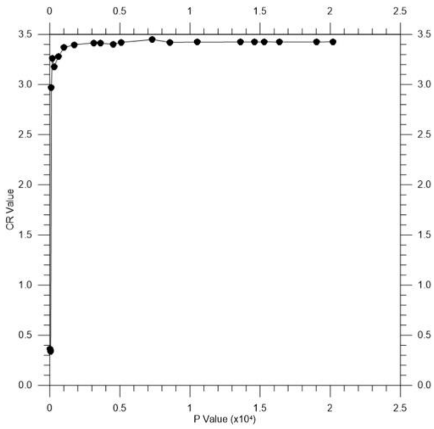
Definition 3.3 Fix an odd prime *p*, let $\boldsymbol{\gamma}$ be a primitive generator of the field $\boldsymbol{GF(p)}^*$, define the cyclotomic class of order 4 as

$$D^i = \{\gamma^{\boldsymbol{i}}, \gamma^{i+4}, \gamma^{i+2\times 4} \cdots, \gamma^{p-5+i}\}, \quad i = 0,1,2,3, \qquad (6)$$

By Chinese Remainder Theorem (CRT), Z_{8p} is isomorphic to $Z_8 \times Z_p$. Define $\boldsymbol{c} = \{\{n\} \times C_n\}$ where $0 \le n < 8$, and $C_n = D^i \cup D^j$, where D^i is defined as in (6). From now on, without loss, we denote $C_n = D^i \cup D^j$ as $C_n = (i,j)$ where $i,j = 0,1,2,3$.

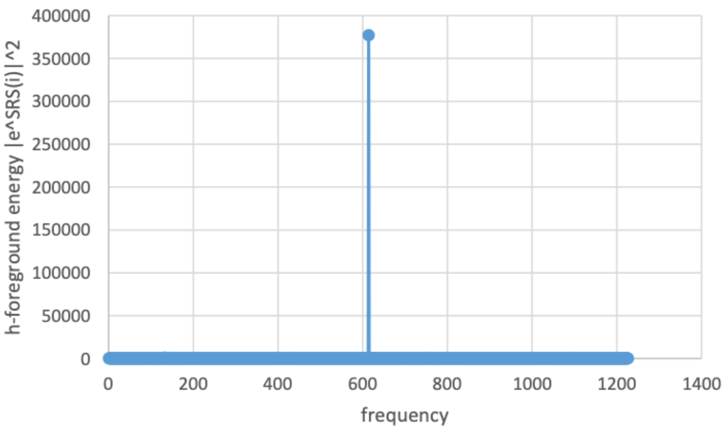
Result 3.4 Let an odd prime p be of form $\frac{n^2+1}{2}$**,** where n is an integer. Then choose $C_0 = (0,3)$, $C_1 = (1,2)$, $C_2 = C_3 = (0,1)$, then $C_4 = (1,2)$, $C_5 = (0,3)$, $C_6 = C_7 = (2,3)$ then the characteristic sequence over the subset $\boldsymbol{c} = \{\{n\} \times C_n\}$ has the asymptotic $CR^{SRS}(4p)$value ~3.42 as shown in Figure 5.

Figure 5: The asymptotic $CR^{SRS}(4p)$ ratio of Parker Sequences of length 8p is about 3.42



Based on Figure 5, we can draw graphs similar to Figure 2. Obviously both CR^{SRS} and $\frac{S}{N_R}$= values the sequences presented in current paper, are significantly higher than the carefully chosen sequence published in [2]

Figure 6: TH sequence with length 1228 with the maximum $|e^{SRS}(614)|^2$ = 376996, the contrast ratio $CR^{SRS}(614)$= 5.6533 and $\frac{S}{N_R(614)}$= 6930.95

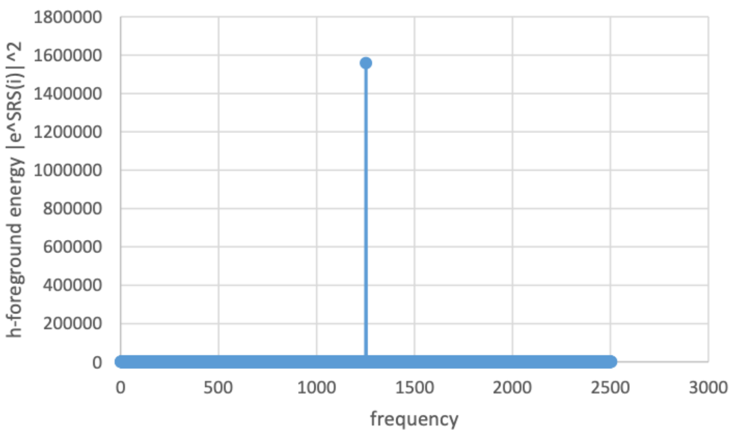


4. Conclusion

In this paper we have demonstrated the close relation between the central contrast ratio CR^{SRS} and traditional Merit Factor problem. Therefore, instead of conducting exhausting search of "ideal" binary sequences, we have provided several classes of binary sequences with high $CR^{SRS}(h)$, or equivalently, h-signal-to-noise ratio $\frac{S}{N_R(h)}$.

In this section, we will summarize the central CR^{SRS} values on several well-known families of binary sequences. All the sequences listed in this section have significantly higher CR^{SRS} and $\frac{S}{N_R}$ values than the results in existing documents.

Figure 7: Parker sequence with length 2504 *with the maximum* $|e^{SRS}(1252)|^2$ = *1557504, the contrast ratio* $CR^{SRS}(1252)$= *3.177 and* $\frac{S}{N_R(1252)}$= *7955.21*



For SRS problem, it does not matter whether the length of the sequence $\boldsymbol{e}$ is even or odd. However, sequences with different forms $\boldsymbol{e} = (\boldsymbol{c} \mid \boldsymbol{c})$, or $\boldsymbol{e} = (\boldsymbol{c} \mid -\boldsymbol{c})$ have different $\boldsymbol{CR^{SRS}}$ values. In particular,

SRS.1 Let $\boldsymbol{e} = (\alpha^{\frac{1}{4}} \mid \alpha^{\frac{1}{4}})$, where α is a Legendre sequence of length q, or a Jacobi sequence of length pq as defined before. Let $\alpha^{\frac{1}{4}}$ be the sequence obtained by cyclically shifting α by a ratio of $\frac{1}{4}$. Then the asymptotic contrast ratio $CR^{SRS}(q)$ for Legendre or $CR^{SRS}(pq)$ for Jacobi is 6.0, when $p = 1$ or $p < q$ are distinct odd primes with p and q close enough.

SRS.2 Let $\boldsymbol{e} = (c \mid c)$, where c is an m-sequence of length $2^n - 1$, Then the asymptotic contrast ratio CR^{SRS} for $\boldsymbol{e}$ is 3.0.

SRS.3 Let $\boldsymbol{e} = (\beta \mid -\beta)$, where β is a TH-sequence of length $2p$. Then the asymptotic contrast ratio CR^{SRS} for $\boldsymbol{e}$ is 6.0.

SRS.4 Let $\boldsymbol{e} = (s \mid -s)$, where s is a Parker sequence of length $4p$, then

SRS.4.1 if p is of form $n^2 + 4$, and s is generated based on cyclotomic classes $\{\{0,2\}, \{1,3\}, \{1,2\}, \{1,2\}\}$, then numerical analysis shows that the asymptotic contrast ratio CR^{SRS} for $\boldsymbol{e}$ is ~2.67.

SRS.4.2 if p is of form $\frac{n^2+1}{2}$, and s is generated based on cyclotomic classes $\{\{0,3\}, \{1,2\}, \{0,1\}, \{0,1\}\}$, then numerical analysis shows that the asymptotic contrast ratio CR^{SRS} for $\boldsymbol{e}$ is ~3.42.

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