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Layers of Mathematical Understanding and Folding Back of Prospective Teachers on Arithmetic Sequence and Series

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Abstract
Mathematical understanding is one of the objectives of learning mathematics, so students need to master it. The success of the process of growing students' mathematical understanding is inseparable from the role of the teacher. Therefore, student teacher candidates must have a complete mathematical understanding structure. This is what prompted him to carry out research to describe the layers of mathematical understanding and folding back prospective teacher students on the material of arithmetic sequences and series. The researcher took 18 sixth-semester students of S1 Mathematics Education at the State University of Malang. A purposive sampling technique was used to determine the participants. So, one subject was chosen in each category of mathematical understanding. Researchers used a descriptive approach with three stages of research procedures including planning, implementing, and drawing conclusions. The instruments used consisted of test questions and interview guidelines. Based on analysis carried out using indicators for each layer of mathematical understanding, information was obtained that subjects with high abilities were able to achieve layering arrangement and perform effective folding back independently. Subjects with moderate abilities were able to reach the observation layer and fold back effectively but needed help. Meanwhile, subjects with low abilities were able to reach the formalization layer but had not been able to fold back effectively. It can be concluded that subjects with better mathematical understanding have a more complete layered structure of mathematical understanding and can fold back independently and effectively.

Keywords: Mathematical understanding, folding back, Pirie Kieren, prospective teachers, arithmetic sequences and series

1. Introduction

Mathematical understanding is an important element in learning mathematics. In addition to being one of the goals of mathematics learning (Andamon & Tan, 2018), mathematical understanding can encourage students to master the material more flexibly, resulting in intellectual satisfaction for students (Hiebert, 1997). Mathematical understanding is also a means that is considered capable of realizing the vision of developing mathematics learning. The vision in question is the role of using mathematics in solving mathematical problems in everyday life and other disciplines (Sumarmo, 2013). Therefore, mathematical understanding is considered an important thing in learning mathematics.

According to Skemp (1987), mathematical understanding is the ability to incorporate appropriate schemes and concept structures during learning. Although Skemp divides mathematical understanding into two types, namely conceptual and procedural understanding, basically the ability to understand concepts and proper procedures is a unity that cannot be separated (Usiskin, 2012). Students' mathematical understanding is certainly not constant. Mathematical understanding can grow dynamically sequentially, multilevel but not linearly, and recursively (Pirie & Kieren, 1994). Pirie & Kieren (1989) categorized the growth process of mathematical understanding into eight main layers including primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. This categorization is henceforth called Pirie Kieren's theory. In addition to the eight main layers, according to Pirie & Martin (2000) in the growth process of mathematical understanding, there is a cognitive process called folding back. Folding back is a cognitive activity that students do to reorganize their understanding when facing a problem by returning to deeper layers of mathematical understanding. (Martin, 2008). So it can be concluded that students do folding back periodically to form good mathematical understanding skills.

Several parameters need to be met so that someone is said to have good mathematical understanding skills. The parameters in question include having the ability to define concepts verbally, give examples and non-examples, present a concept in various representations, make connections between concepts, and use procedures appropriately and accurately (Kilpatrick et.al., 2001). So that students are said to have achieved good mathematical understanding if they fulfill some of these parameters.

The success of students in achieving good mathematical understanding is influenced by the role of the teacher (Armelia et al., 2019). This is because teachers are the spearhead of achieving educational goals (Sukmawati, 2019). Therefore, teachers should have a good mathematical understanding as well. Hatta (2018) mentioned that the ability to master the material and basic concepts of the material to be taught is something that teachers must have. Therefore, teachers' mathematical understanding is one of the elements that must be fulfilled to master the material broadly and deeply. This ability refers to the teacher's professional competence (Sukmawati, 2019). Professional competence is one of the competencies that must be possessed by teachers based on Law Number 14 of 2005 Article 8. This is what underlies that prospective teachers need to prepare these competencies early on (Lubis, 2018). Therefore, it can be concluded that it is important for prospective teachers to have a good mathematical understanding (Donuata & Pratama, 2021). In other words, prospective teachers must have a complete mathematical understanding of layer structure (Sagala, 2017).

Based on the facts of the existing literature, there are still many prospective teachers who do not have good mathematical understanding skills. In research conducted by Sukmawati (2019), more than 50% of prospective teachers (out of 90 subjects) have not maximally fulfilled professional competencies. The professional competence in question is the ability of prospective teachers to understand the material. In other words, teachers do not have a good mathematical understanding. As a result, many of them still teach by looking at notes, jumping around, and even misconceptions. The lack of mathematical understanding of prospective teachers was also found in the research of Musyrifah et al (2022). Student teacher candidates still experience conceptual ontological obstacles, namely conditions when students still do not understand the concepts used. The same fact was found in the research of Hoiriyah et al (2019) and Tamba et al (2022), namely that more than 60% of prospective teacher students failed the test that tested mathematical understanding. Based on the exposure to the facts of the literature, it can be concluded that the mathematical understanding of prospective teachers is still low. This condition results in their failure to fulfill professional competencies. So it is not surprising that many prospective teachers have sufficiently fulfilled pedagogical, personality, and social competencies, but are still lacking in professional competencies. professional competence (Witoni & Zakaria, 2020).

The literature facts found by the researcher are not much different from the field facts that the researcher obtained. To support the facts of the literature that have been obtained, researchers made observations to prospective teacher students in semester VI of the undergraduate mathematics education study program at the State University of Malang. Researchers made observations when students attended lectures in math game classes. The output of the course is that students can design math games that can be implemented in innovative mathematics learning. In the observations made, it was found that students were able to design math games but the conceptual problems underlying the games were still not well mastered. This condition supports the fact that prospective teachers can design lessons but still do not have good concept readiness. In other words, the mathematical understanding of prospective teachers is still lacking. Therefore, based on the facts of the literature supported by the facts of the field, it can be concluded that there is a need for a study related to mathematical understanding for prospective teacher students.

Several studies have been conducted studies related to the mathematical understanding of prospective student teachers. Arifin & Aprisal (2020) and Tamba et al (2022) are examples of researchers who conducted mathematical understanding studies specifically on the conceptual abilities of prospective teacher students. The research was conducted to analyze overall conceptual ability based on the indicators used. Slightly different from these studies, Sagala (2016) conducted a study related to the profile of students' mathematical understanding of derivative material with a focus on differences in folding back based on gender. In addition, there are also studies conducted by Sagala (2017) and Donuata & Pratama (2021). Both studies have the aim of describing the layers of mathematical understanding of prospective teacher students using Pirie Kieren's theory. Almost similar to, Muliawati (2020) in her research also focused on studying the layers of mathematical understanding in prospective teacher students using Pirie Kieren's theory but with the focus of the subjects being students with the middle ability category only.

Based on some of these previous studies, researchers found that research on prospective teachers' mathematical understanding is still limited. Some of them only discuss the conceptual understanding of prospective teachers (Arifin & Aprisal, 2020; Tamba et al, 2022). Whereas mathematical understanding not only includes conceptual understanding but also procedural understanding (Usiskin, 2012). Meanwhile, several other studies attempted to describe the mathematical understanding of prospective teachers using Pirie Kieren's theory but were limited to certain focuses such as gender (Sagala, 2016) or certain ability levels (Muliawati, 2020). Good mathematical understanding must be possessed by all prospective teachers considering that this ability is one of the competencies that teachers must master (Lubis, 2018). Based on that, the researcher decided to conduct a study to describe the layers of mathematical understanding and fold back of prospective teacher students on subjects with low, medium, and high abilities. This research is important to do to prepare prospective teachers who fulfill professional competencies well. The material used in this study is the material of arithmetic sequences and series. Sequences and series are important materials for students to master (Bell, 2016). However, the fact is that students' understanding of the material still needs to be improved (Damayanti et al, 2022; Masuda et al, 2021; Putri Khairani et al, 2021). For this reason, researchers chose the material of rows and series as the concept involved in this study.

2. Methods

In this study, researchers used descriptive qualitative research methods. The method was chosen to help achieve the research objectives, namely describing the layers of mathematical understanding and folding back of prospective teacher students on the material of arithmetic rows and series. The research procedures carried out refer to the stages of descriptive qualitative research according to Setyaningsih (2022) which includes the planning stage, the implementation stage, and the conclusion stage.

In the planning stage, researchers conducted literature reviews and observations to see the mathematical understanding abilities of prospective teacher students. At this stage, the researchers also designed several research instruments including test questions, rubrics, and interview guidelines. At the implementation stage, researchers gave test questions to 18 undergraduate students of Mathematics Education, at the State University of Malang semester VI. After that, the researcher conducted purposive sampling to determine the research subject. Researchers took one each from students with high, medium, and low mathematical understanding. Then the researcher conducted interviews with the three subjects. The last stage is concluding. The data sources used in this research are test questions and video interviews. Based on the answers to the test questions, the researchers described the layers of mathematical understanding and folding back by referring to the indicators in each layer of Pirie Kieren's mathematical understanding. Triangulation was done by comparing students' answers with the results of interviews (video interview analysis). The indicators of Pirie Kieren's mathematical understanding layers used are the results of adaptation from Putri & Susiswo's research (2020) which are stated in Table 1 below.

Table 1: Indicators of Pirie Kieren's Levels of Mathematical Understanding

Level of Mathematical Understanding	Indicator
Primitive Knowing	Students have the prior knowledge needed to build a concept (Students try to understand the definition by involving definitions or representations of prior understanding)
Image Making	Students do the activity of creating a mental picture of a topic by understanding prior knowledge and using it for new knowledge.
Image Having	Students understand a topic without having to perform an activity or example that triggers it.
Property Noticing	Students can connect and combine different aspects of the topic to form a distinctive feature of the constructed picture
Formalising	Students can generalize the characteristics obtained on the previous layers into formal concepts
Observing	Students can use their concepts
Structuring	Students can organize and connect existing concepts into a proven theory
Inventising	Students can create new questions into new concepts.

3. Results & Discussion

In carrying out the research, researchers gave test questions to 18 sixth-semester students of the S1 Mathematics Education study program at the State University of Malang. The test questions consisted of seven items. These questions were used to measure students' mathematical understanding based on indicators of mathematical understanding. Table 2 below is a summary of student test results.

Table 2. Mathematical Comprehension Test Results

Number of Students	Maximum Value	Minimum Value	Average
18	93	17	45,33

Based on the test results, researchers grouped students into 3 categories, namely high, medium, and low groups. The provisions on which the grouping is based are based on the scale made by Tsurayya & Nur (2021). Table 3 below shows the level of mathematical understanding based on these provisions.

Table 3. Mathematical Comprehension Level

Category	Terms	Number of Students	Percentage
High	> 60,92	4	22,22%
Medium	32,07 ≤ Value ≤ 60,92	7	38,88%
Low	< 32,07	7	38,88%

Based on the categorization in Table 3, the researchers selected one student each to be the research subject. The researcher chose GI, LA, and AT as research subjects. GI is a student with a high mathematical understanding category. LA is a student with moderate mathematical understanding category. Meanwhile, AT is a student with a low mathematical understanding category. The researcher analyzed the answer sheets of the three subjects. To support the data, researchers conducted unstructured interviews with the three subjects to explore further information to describe the layers of mathematical understanding and folding back that they do. The following is an explanation of the layers of mathematical understanding and folding back of the three subjects.

3.1 Layers of Mathematical Understanding and Folding Back of GI

Based on the test answers and interviews conducted, the researcher obtained the fact that GI has an almost complete mathematical understanding of the arithmetic sequence material, namely up to the structuring layer. This condition is shown by the subject being able to use the formal concept of arithmetic sequence to prove $2^{a_1}, 2^{a_2}, 2^{a_3}, \ldots$ is a geometric sequence when $a_1, a_2, a_3, \ldots$ is an arithmetic sequence. In the proof strategy, GI was also able to make a generalization that it will be valid until the $n + 1$ term. This shows that, GI realized that the proof must be proven true not only by the first two terms. However, it is true until the nth term. So that the proof given can be said to be valid.

Figure 1 Snippet of GI Answers

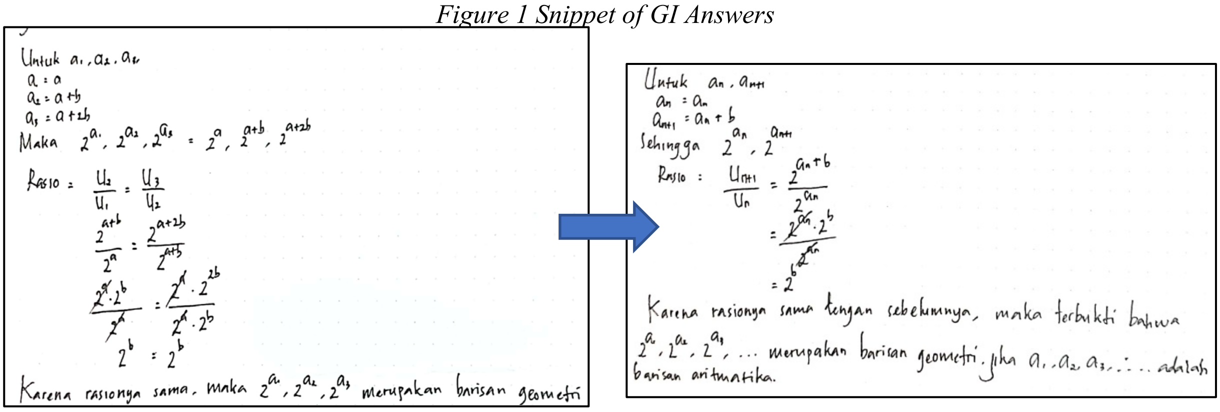


Figure 1 above, shows that GI has fulfilled the mathematical understanding indicator of the structuring layer. That is, being able to organize and connect existing formal concepts into a proven theory. To explore the thinking process carried out, the researcher interviewed the GI subject. The following are excerpts of interviews with GI subjects.

R : *"In question 6, there is a command to prove that $2^{a_1}, 2^{a_2}, 2^{a_3}, \dots$ is a geometric row $a_1, a_2, a_3, \dots$ is an arithmetic row. What is your thinking strategy to solve this problem?"*

GI.1 : *"First I tried for a_1, a_2 and a_3. So from the smallest first Ma'am. Is it a geometric row"*

R : *"You wrote that, $a_1 = a$, $a_2 = a + b, a_3 = a + 2b$ and so on, why did you write it that way?"*

GI.2 : *"Because $a_1, a_2, a_3, \dots$ is an arithmetic sequence. We know that the first term can be expressed as a, the second term , the third term $a + 2b$ and so on"*

R : *"Okay, why did you also check up to U_n and U_{n+1}?"*

GI.3 : *"So that the proof applies in general, not just a_1, a_2, a_3"*

R : *"Okay fine. Next I want to ask, so far do you have any critical questions related to arithmetic series and sequence"*

GI.4 : *"No Ma'am"*

The interview above shows that GI experienced folding back. Statement GI.1 shows that GI returned to the primitive knowing layer to confirm whether the given sequence is a geometric sequence by utilizing the definition of arithmetic and geometric sequence. The next action taken was to conduct a trial by utilizing $a_1, a_2, a_3, \dots$ as an arithmetic sequence. This shows that GI moves to the image-making layer. Statement GI.2 shows that GI started working in the formalizing layer. GI utilized the properties and formal form of the formula for the nth term of the arithmetic sequence to confirm the trials conducted. Meanwhile, statement GI.3 shows that GI entered the property noticing layer. GI understands that arithmetic and geometric series have the property of being valid up to the nth term, not just stopping at the first three terms. Meanwhile, statement G1.4, shows that GI has not been able to develop his understanding of the inventing layer. The subject has not been able to come up with critical questions that can bring up new concepts in the arithmetic sequence material.

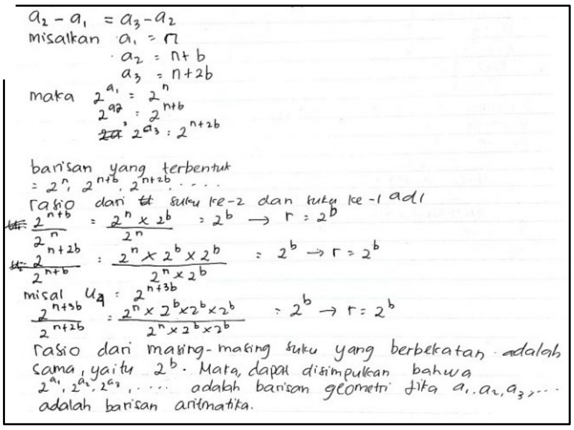
In addition, by obtaining information related to the description of GI's mathematical understanding layer, in the interview, the researcher can confirm that GI has provided clear and effective responses and reasons during the folding back process. GI was also able to fold back without any help from the researcher. Folding back that GI did only happened once, namely with the flow of primitive knowing → image making → property noticing → formalizing.

3.2 Layers of Mathematical Understanding and Folding Back of LA

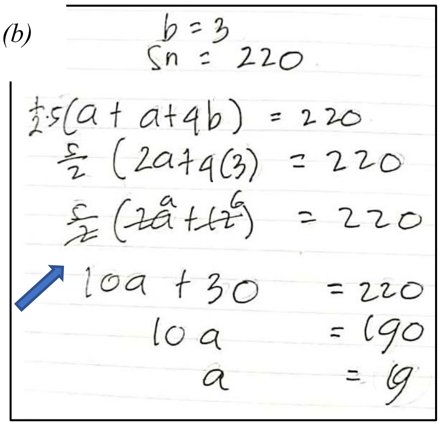
Based on the test answers and interviews conducted, the LA subject has a mathematical understanding up to the observing layer. The researcher also obtained information that LA had problems in the structuring layer of understanding. The following is a snippet of LA's test answer which is one of the bases for this statement.

Figure 2 Snippet of LA Answers

(a)



(b)



As seen in Figure 2.a, LA tries to prove 2^{a_1}, $2^{a_2}, 2^{a_3}, \dots$ is a geometric row when $a_1, a_2, a_3, \dots$ is an arithmetic row. However, unlike GI, LA only proved the statement up to the first three terms. To explore the thinking process, the researcher interviewed with LA subject. The following is an excerpt from the interview with LA subject.

R	:	*"Try to explain what you understand from the problem, prove that* 2^{a_1}, $2^{a_2}, 2^{a_3}, \dots$ *is a geometric row when* $a_1, a_2, a_3, \dots$ *is an arithmetic row"*
LA.1	:	*"Supposedly, the powers are arithmetic lines, so we prove the powers are arithmetic lines and the two power lines formed are geometric lines"*
R	:	*"Try, read again, what information is given and what do you want to prove?"*
LA.2	:	*'The information* $a_1, a_2, a_3, \dots$ *is an arithmetic sequence and it will be proved that* 2^{a_1}, $2^{a_2}, 2^{a_3}, \dots$ *geometry"*
R	:	*"When we prove something, we usually use the information given, based on the information given* $a_1, a_2, a_3, \dots$ *is an arithmetic sequence,what does that mean then?"*
LA.3	:	*"The difference is the same, so I wrote it down for example* $a_1 = n$, $a_2 = n + 2b$*"*
R	:	*'Did you prove it only up to* U_4*?"*
LA.4	:	*"Oh yes...it should apply and so on"*

The interview above shows that LA experienced folding back. LA.1 statement shows that LA initially did not understand the meaning of the problem. So that he could not distinguish between information and statements that must be proven. However, with the help of questions, LA could finally answer firmly as stated in statement LA.2. Statement LA.3 shows that LA experienced folding back to the primitive knowing and formalizing layers. LA utilized the definition of line arithmetic and used its formal form to prove the statement. However, LA did not realize that the proof must apply up to the nth term. Based on LA.4 statement, LA assumed that when he had proved up to the 4th term, it means that it applies forever. This condition supports the statement that LA failed to fulfill the structuring layer indicator.

Based on the analysis of the previous interview, it was found that LA could provide clear responses and reasons during the folding back process, although it needed help from the researcher by asking more specific questions. LA's folding back occurred not only once, namely with the flow of primitive knowing → formalizing → property noticing

Besides trying to describe the folding back done by LA, the researcher also conducted further analysis to see the layers of LA's mathematical understanding. For this reason, the researcher analyzed LA's answer as shown in Figure 2. b. In the picture, it can be seen that LA used the concept of an arithmetic sequence to solve problem number 7. Problem number 7 specifically contains illustrations of problems that can be solved by applying the concept of arithmetic sequence. In the answer, it is clear that LA wrote the concept correctly, but there was an error operating the algebraic multiplication shown by the blue arrow. To explore further information, the researcher interviewed to confirm LA's answer. The following is an excerpt from the interview.

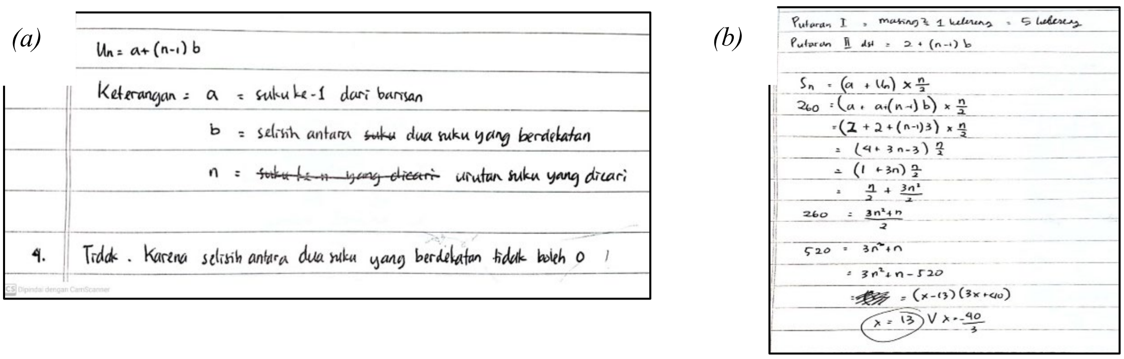
R : *"Explain what you understand from question no. 7"*
LA.5 : *"At first I misunderstood the question, then I changed it. Right, in the first round there were five people each taking 1, followed by the second round there were five people each taking 2, 5, 8, ... and so on, then continued the third round there were five people each taking three at odds with the previous person"*
R : *"Since there will be 265 marbles in total, how many have been taken in the first and second rounds?"*
LA.6 : *"Let me count... there are 45"*
R : *"Then for the third round, how much does he have to collect?"*
LA.7 : *"220"*
R : *"How do you determine the marbles taken by the first person in the third round?"*
LA.8 : *"Hmmm... arithmetic sequence because the difference is taken equally"*
R : *"In your answer on the third line you operate* $\frac{5}{2}$ $(2a + 12)$ *by dividing 2 and* $(2a + 12)$ *and getting* $10a + 30$*. Why?"*
LA.9 : *"I think I wrote it wrong at the beginning of the calculation, I didn't correct it so it was wrong. It should be* $5a + 30$*".*

Based on the interview and the answers in Figure 2. b, it can be confirmed that LA has fulfilled the observing mathematical understanding indicator. LA can apply the concept of arithmetic sequence in solving problems. This is confirmed in LA.8 statement. Students can also write and use procedures appropriately in solving problems related to arithmetic sequences. However, the final result is still not correct due to inaccuracy in the algebraic multiplication operation performed. This can be confirmed in LA.9 statement.

3.3 Layers of Mathematical Understanding and Folding Back of AT

Based on the test answers and interviews conducted, it was found that AT had a mathematical understanding of arithmetic sequence up to the formalizing layer. This is shown by AT being able to combine and connect various aspects to form a special characteristic of an arithmetic sequence. However, students have not been able to apply the concept appropriately. In other words, they have not been able to fulfill the observing indicator. The following is a snippet of AT's test answer which is one of the bases for this statement.

Figure 3 Snippet of AT Answers



Based on Figure 3, it can be seen that AT has been able to write the formula form of arithmetic sequence and row correctly. So it can be categorized as fulfilling the indicators of the formalizing layer. However, AT failed to apply the concept of arithmetic sequence well. In question 4, when instructed to decide whether 1,1,1,1 ... is an arithmetic sequence, AT failed to give a decision. As shown in Figure 3. a, AT decided that 1,1,1,1, ... is not an arithmetic sequence with the reason that the difference is 0. To explore further information, the researcher interviewed AT to confirm his answer. The following is an excerpt from the interview with AT.

R : "You stated that 1,1,1,1 ... is not an arithmetic sequence because the difference between two adjacent terms in an arithmetic sequence cannot be 0, what do you think an arithmetic sequence is?"

AT.1 : "Rows that have the same difference between terms"

R : "What about the difference between terms 1,1,1,1 ... is it the same?"

AT.2 : "The same is 0, I answered that it is not an arithmetic sequence because I thought that if the difference is 0, it means that all the terms are equal to U_1. So I thought they would be singular instead of a number sequence"

R : "Then what do you think the row is?"

AT.3 : "It has a pattern and the difference cannot be the same"

R : "Okay, then for question no.7, based on your answer, it says in rounds two and three you used the Sn formula, even though the rules are different, why?"

AT.4 : "For the second round of U_1 I filled in 2 because the first person immediately took two marbles"

R : "Then why did you use the formula Sn"

AT.5 : (Thinking for a moment) "Sn is to find the number of times the ball is taken, in other words, I want to know how many rounds they play. Because the number of marbles is 260, through the calculation, it is found that there will be 13 takes"

Based on the interview excerpts, it can be seen that although AT can write the formal form of an arithmetic sequence, she cannot apply it well in solving the given problem. Therefore, it can be concluded that AT fulfills the indicators of the formalizing layer but fails in the observing layer. This fact can be proven from the statements she gave in AT.2 and AT.5. In Figure 3. a, it is written that AT can write the formula Un Likewise, in Figure 3. b AT can write the formula Sn correctly. However, from the statement AT.2 it is clear that AT cannot connect the formal form of the nth term of the arithmetic sequence to recognize 1,1,1,1 ... as an arithmetic sequence. The same thing happened in statement AT.5. AT failed to understand the use of elements in the formal form of arithmetic sequence (Sn). The subject considered n as the number of rounds to be performed. Whereas n states the order of people. This supports the opinion that AT successfully reached the formalizing layer, but failed at the observing layer in the growth process of her mathematical understanding.

In the process of building her mathematical understanding, AT folded back. This condition is shown in the AT.1 statement. AT initially folded back with the stimulus from the researcher's question. AT returned to the primitive knowing layer to recall the definition of arithmetic rows and series. It is implied in AT.1 statement that to categorize a sequence as an arithmetic sequence, AT considers the difference between the tribes to be the same. When the researcher asked a question related to the difference of 1,1,1,1 ... how much, AT was able to say the difference was 0. However, as implied in her statement in AT.2, AT denied the sequence as an arithmetic sequence. This condition shows that AT tried to fold back, but it did not succeed in helping her to solve the given problem. However, after getting help from specific questions given by the researcher.

3.4 Other Findings

After going through the process of analyzing the answers and interviews with the three subjects, the researcher found unique facts during the observation process. Although they were able to have mathematical understanding up to the formalizing, observing, and structuring layers on procedural understanding questions, the three subjects experienced obstacles in their primitive knowing layer. For example, for the subject GI. GI had difficulty when building his conceptual understanding. At first, GI failed to understand the sequence as a function. GI could not distinguish between number patterns and number lines. GI wrote on her test answer sheet that a sequence is a set of numbers that has a certain pattern. Therefore, the researcher interviewed to confirm this answer. The following is an excerpt of the interview transcript with subject GI.

R : "You wrote that a row is a set of numbers that have a certain pattern, does a row have to have a pattern?"

GI.4 : "After I remember, a row doesn't have to have a pattern, because a row is a function so it shouldn't need to have a pattern, except for arithmetic and geometric rows which have accompanying rules/patterns"

R : "You wrote that number terms are the numbers that make up the sequence, with the new definition that you understand, does your understanding of the sequence terms still remain the same?"

GI.5 : "e.e.e. one second. It looks like it's still there"

R : "You wrote that an arithmetic sequence is a sequence whose terms have a difference of $U_2 - U_1$*, what does that mean?"*

GI.6 : "Actually, I want to write that an arithmetic sequence is a sequence with the same difference in adjacent terms, but I'm confused about writing it"

R : "Then in question no. 2, there is a sequence -2,3,1,4, ... you determine the fifth term is 4 through the process of finding a pattern. Is the answer single, if for example I answer 1000 is it okay?"

GI.7 : "It's okay because the row doesn't have to have a pattern"

Based on the interview transcript, it can be concluded that GI has fulfilled the primitive knowing indicator, which is that students can understand the definition of the row well. This is supported by GI's ability to understand the definition of an arithmetic sequence. This condition is shown in statements GI.4 and GI.6. Although the definitions were delivered using their language (informal). In addition, GI was also able to understand the concept of the sequence by giving the right response to the statement given by the researcher as shown in statement GI.7. This shows that GI began to move to the image-making layer. As stated in the previous discussion, GI also managed to understand the concept of arithmetic rows and series until the structuring layer.

However, there is a unique phenomenon that the researcher obtained. Statement GI.5, shows that the subject could not connect the definition of a line with the line terms. GI understood the row as a function but failed to understand the row term as an element in the function, namely the function value. As a result, in statement GI.6, it can be seen that GI tried to identify an arithmetic sequence with the difference of adjacent terms being the same. GI conceptually understands that what is meant by the difference in tribes is the difference in the values of adjacent tribes, but GI cannot restate the concept verbally with the right definition. This condition raises an indication that GI subjects have primitive knowing abilities that are less elaborated. So the ability to connect the prerequisite concepts with the core concepts of the discussion is still lacking. The same thing also happened to LA and AT subjects. The two subjects did not understand that the row is a function. LA was only able to recognize the row as a collection of consecutive numbers and had no pattern. Meanwhile, AT understood the sequence as something that must have a pattern.

Mathematical understanding can grow following eight layers including primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing (Pirie & Kieren, 1989). Student teachers as future teachers certainly need to have a good mathematical understanding. This is regulated in Law No. 14 of 2005 Article 8 which states that one of the mandatory competencies that teachers must have is professional competence. One of the indicators of professional competence is the achievement of a good understanding of the mathematics material presented. Referring to this foundation, the researcher examined the layers of mathematical understanding of prospective teacher students as well as folding back carried out based on predetermined indicators. Table 4 is a summary of the data obtained.

Table 4. Summary of Research Results

Category	Comprehension Layer	Folding Back
High	Structuring	Effective with independence
Medium	Observing	Effective with help
Low	Formalizing	Ineffective

Based on Table 4, the researcher obtained information that students with high mathematical understanding (GI) have been able to reach the structuring mathematical understanding layer. These students have also been able to fold back effectively and independently. GI folded back once with the flow of primitive knowing → image making → property noticing → formalizing without help from the researcher. GI has also given appropriate and effective responses during folding back. This can be confirmed in the interview that has been conducted. Such folding back results according to Susiswo (2014) are called continuous effective folding back. The subject can apply understanding to deeper layers and work on the next outer layer without folding back again.

Meanwhile, the researcher also obtained information that the subject with moderate mathematical understanding (LA) had mathematical understanding up to the observing layer. LA also managed to fold back effectively even though it still needed help with questions from the researcher. LA managed to do effective folding back twice with the flow of primitive knowing → formalizing → property noticing. Such folding back results, according to Susiswo (2014), are referred to as pseudo-effective folding back. This is supported by the information that the subject can apply understanding at a deeper layer and work on the next outer layer by folding back to property noticing.

In addition to describing the understanding of subjects with high and medium mathematical understanding, researchers also described the mathematical understanding of subjects with low categories (AT). Based on Table 4, it is obtained that AT has reached the formalizing layer of understanding. However, AT failed at the observing layer because he failed to fold back. AT is confirmed to have tried to return to the primitive knowing layer but failed to work at that layer. Such folding back results according to Susiswo (2014) are called ineffective. The subject failed to work on deeper layers to build an understanding of the outer layers.

In addition to the findings summarized in Table 4, the researcher also found that the three subjects had not been able to elaborate on their primitive knowing abilities. For example, GI failed to connect the concepts of sequence and function. Although GI could understand a topic well, he was unable to find connections between one concept and another. According to Martin & Tower (2016), this condition can result in a person not being able to understand mathematics in an integrated and comprehensive manner. This ability is also one of the indicators of professional competence that must be possessed by a teacher, namely having a broad and deep understanding of concepts (Sukmawati, 2019). Therefore, prospective teacher students should not only have a complete understanding structure of a topic but can develop it for other interconnected topics. So that math can be understood as a whole and integrated.

4. Conclusions and Suggestions

Good mathematical understanding is an ability that students must have. Teachers are one of the factors that determine the success of students in developing their mathematical understanding. Thus, teachers should have good professional competence with one of the indicators being a broad and deep understanding of the concepts they teach. Therefore, prospective teachers need to have a complete mathematical understanding structure.

Mathematical understanding ability can be categorized into high, medium, and low groups. Based on the research that has been carried out, subjects with high-category mathematical understanding can reach the structuring layer of mathematical understanding and can fold back effectively and independently. Subjects with moderate mathematical understanding can reach the observing layer of mathematical understanding and can do folding back effectively but need to get help from researchers. While subjects with low mathematical understanding can reach the formalizing layer but have not been able to fold back effectively. It can be concluded that subjects with better mathematical understanding have a more complete layered structure of mathematical understanding and can fold back independently and effectively. In addition to these findings, other interesting findings were obtained. Namely, it was found that the three subjects had abilities in elaborated primitive knowing. It can be concluded that in addition to a complete structure of mathematical understanding layers, prospective teacher students need to have mature abilities in primitive knowing. This condition is indicated by being able to link the prerequisite concepts to the concepts to be learned. In other words, they can elaborate on the ability of primitives to know well. Therefore, for further research, it is necessary to study the role of primitive knowing in the success of good mathematical understanding and the elaboration ability of prospective teacher students on their initial abilities.

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References

Andamon, J. C., & Tan, D. A. (2018). Attitude And Performance in Mathematics of Grade 7 Students Article in. *International Journal of Scientific & Technology Research*, *7*(8). www.ijstr.org

Arifin, S., & Aprisal, A. (2020). Analisis Tingkat Pemahaman Konsep Statistika Mahasiswa Calon Guru Menggunakan Two Tier Test Berbasis Online. *Delta: Jurnal Ilmiah Pendidikan Matematika*, *8*(2), 201. https://doi.org/10.31941/delta.v8i2.1059

Armelia, D., Prihatin, I., Susiaty, U. D., Studi, P., Matematika, P., & Pgri Pontianak, I. (2019). Pengembangan Media Pocket Book Berbasis Discovery Learning Terhadap Kemampuan Pemahaman Matematis. *Jurnal SAP*, *3*(3).

Bell, C. J. (2016). Lining Up Arithmetic Sequences. *Mathematics Teaching in the Middle School*, *17*(1), 34–39. https://doi.org/10.5951/mathteacmiddscho.17.1.0034

Damayanti, N., Bina Widya, K. K., & Baru, S. (2022). *Mosharafa: Jurnal Pendidikan Matematika Analisis Kemampuan Pemecahan Masalah Matematis Siswa SMA pada Materi Barisan dan Deret Geometri*. *11*(1). http://journal.institutpendidikan.ac.id/index.php/mosharafa

Donuata, I. G., & Pratama, F. W. (2021). Lapisan Pemahaman Konsep Mahasiswa Calon Guru Matematika dalam Menyelesaikan Soal Logaritma. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, *10*(3), 1541. https://doi.org/10.24127/ajpm.v10i3.3701

Hoiriyah, D., Tarbiyah, F., Keguruan, I., & Padangsidimpuan, I. (2019). Analisis Kemampuan Pemahaman Konsep Matematis Mahasiswa. In *Jurnal Ilmu-ilmu Pendidikan dan Sains* (Vol. 7).

Hatta, M. (2018). *Empat Kompetensi Untuk Membangun Proffesionalisme Guru*. Sidoarjo: Nizamia Learning Center.

Hiebert, J et al. (1997). *Making Sense: Teaching and Learning Mathematics with Understanding. Portsmouth, NH: Heinemann.*

Kilpatrick, Jeremy., Swafford, Jane., Findell, & Bradford. (2001). *Adding It Up: Helping Children Learn Mathematics*. National Academy Press.

Lubis, H. (2018). Kompetensi Pedagogik Guru Profesional. *Best Journal,* 1(2): 16-19.

Martin, L. C. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren Theory. *Journal of Mathematical Behavior*, *27*(1), 64–85. https://doi.org/10.1016/j.jmathb.2008.04.001

Martin, L. & Jo, T. 2016. Folding Back, Thickening and Mathematical Met-Befores. *Journal Mathematics Behavior*, 46: 89-97.

Masuda, A., Sugeng Pambudi, D., & Murtikusuma, R. P. (2021). Analisis Penalaran Matematis Siswa SMA Kelas XI dalam Menyelesaikan Soal Barisan dan Deret Aritmetika Ditinjau dari Gaya Belajar Honey-Mumford. *Jurnal Riset Pendidikan dan Inovasi Pembelajaran Matematika* (*JRPIPM*), 5(1): 56-68.

Muliawati, N. E. (2020). Lapisan Pemahaman Mahasiswa Calon Guru Matematika Dengan Tipe Middle Ability Dalam Menyelesaikan Soal Pembuktian Grup Berdasarkan Teori Pirie-Kieren. *Jurnal Edukasi Matematika Dan Sains*, *8*(2),157. https://doi.org/10.25273/jems.v8i2.7592

Musyrifah, E., Afgani Dahlan, J., Cahya, E., & Hafiz, M. (2022). Analisis Learning Obstacles Mahasiswa Calon Guru Matematika Pada Konsep Turunan. *Fibonacci: Jurnal Pendidikan Matematika Dan Matematika*, *8*(2), 187. https://doi.org/10.24853/fbc.8.2.187-196

Pirie, S., & Thomas, K. (1989). A Recursive Theory of Mathematical Undestanding. *For the Learning of Mathematics.* 9(3):7-11.

Pirie, S., & Thomas, K. (1994). Growth in Mathematical Understanding: How can We Characterise it and How can We Represent it? *Educational Studies in Mathematics*. 26: 165-190.

Pirie, S., & Lyndon, M. 2000.The Role of Collecting in the Growth of Mathematical Understanding.Mathematics Education Research Journal. 12(2):127-146an We Represent it? *Educational Studies in Mathematics.* 26: 165-190.

Putri Khairani, B., Maemunah, & Roza, Y. (2021). Analisis Kemampuan Pemahaman Konsep Matematis Siswa Kelas XI SMA/MA Pada Materi Barisan dan Deret. *Jurnal Cendekia:* 05(02), 1578–1587.

Putri, R. A., & Susiswo. (2020). Analysis of layer of primitive knowing of high school students in linear function material: A study of application of student activity sheets based on Pirie Kieren theory. *AIP Conference Proceedings*, *2215*. https://doi.org/10.1063/5.0000525

Sagala, V. (2016). Profil Lapisan Pemahaman Konsep Turunan Fungsi dan Folding Back Mahasiswa Calon Guru Matematika Berdasarkan Gender. *Jurnal Ilmiah Soul Math*, 4(5): 217-263.

Sagala, V. (2017). Struktur Lapisan Pemahaman Konsep Turunan Fungsi Mahasiswa Calon Guru Matematika. *Jurnal Didaktik Matematika,* 4(2): 125-135.

Setyaningsih, V.P., dan Dani, F. (2022). Analisis Kemampuan Pemecahan Masalah Matematis Siswa SMP Pada Materi Persamaan Garis Lurus. *Prisma*, 11(1): pp 10-20. https://doi.org/10.35194/jp.v11i1.2048

Skemp, R. (1987). *The Psychology of Learning Mathematics*. New Jersey: Lawrence Erlbaum Associates, Inc.

Sukmawati, R. (2019). Analisis kesiapan mahasiswa menjadi calon guru profesional berdasarkan standar kompetensi pendidik. *Jurnal Analisa*, *5*(1), 95–102. https://doi.org/10.15575/ja.v5i1.4789

Sumarmo, Utari. (2013). *Berpikir dan Disposisi Matematik serta Pembelajarannya*.Bandung : FMIPA UPI.

Susiswo. 2014. *Folding Back Mahasiswa dalam Menyelesaikan Masalah Limit.* Disertasi tidak diterbitkan. Malang:FMIPA UM.

Tamba, K. P., Appulembang, O. D., & Listiani, T. (2022). Korelasi antara Keyakinan Belajar dan Pemahaman Konseptual Kalkulus pada Calon Guru Matematika. *JNPM (Jurnal Nasional Pendidikan Matematika)*, *6*(1), 20. https://doi.org/10.33603/jnpm.v6i1.5315

Undang-undang Nomor 14 Tahun 2005 Pasal 8 Tentang Guru dan Dosen. Presiden Republik Indonesia.

Usiskin, Z.2012. What Does it Mean to Understand Some Mathematics?. *Makalah disajikan pada Twelfth International Congress on Mathematical Education*, Seoul, 08-15 Juli.

Witoni, A., & Zakaria. (2020). *Analisis Kompetensi Guru Matematika SMK Negeri di Kabupaten Bengkulu Selatan*. https://ejournal.unib.ac.id/index.php/manajerpendidikan